The Syndrome Decoding in the Head (SD-in-the-Head) Signature Scheme

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Round-2 Design Updates

Choice of the Framework

SDitH v2 - Design Rationale

- **Conservative security**: signature scheme for which the security relies on the hardness of solving ***fully random unstructured*** instances of the syndrome decoding (SD) problem.

  SD Problem: Given a matrix H and a syndrome y, find x such that

  $Hx = y$ and x has w non-zero coordinates.

- **Design Choice**: Signature scheme built upon the ***MPC-in-the-Head*** paradigm, which provides a generic way to build a secure scheme from a hard problem.

SDitH v2 - Design Rationale

- **SD Parameters**:

  - The hardest instances:

    Unique solution, close to the Gilvert-Varshamov frontier

  - choice of the SD field

    - SDitH v1: $GF(251)$ and $GF(256)$
    - SDitH v2: $GF(2)$

🔎 *Motivation to choose GF(2)*:

More conservative security assumption

Hard problem easier to arithmetize

SDitH v2 - Design Rationale

- **MPC-in-the-Head paradigm**: possible existing frameworks
  - SDitH v1: rely on the framework using linear broadcast-based MPC
  - SDitH v2: ***two new MPCitH frameworks*** since the previous NIST deadline:

VOLE-in-the-Head	and	**TC-in-the-Head**
(summer 2023)		(fall 2023)

[BBD+23] Baum, Braun, Delpech, Klooß, Orsini, Roy, Scholl. Publicly Verifiable Zero-Knowledge and Post-Quantum Signatures From VOLE-in-the-Head. Crypto 2023.

[FR25] Feneuil, Rivain. *Threshold Computation in the Head: Improved Framework for Post-Quantum Signatures and Zero-Knowledge Arguments*. Journal of Cryptology, 2025.

SDitH v2 - Design Rationale

Instance	*Trade-off*	*VOLEitH*	*TCitH*
L1 - SD over GF(2)	Short	3 705 B	4 271 B (+15%)
	Fast	4 484 B	5 509 B (+23%)
L3 - SD over GF(2)	Short	7 964 B	8 426 B (+6%)
	Fast	9 916 B	11 374 B (+15%)
L5 - SD over GF(2)	Short	14 121 B	15 618 B (+11%)
	Fast	17 540 B	19 968 B (+14%)

For SD: Between 5-25% of difference, in favor of VOLEitH

Trade-off definition:
- « Short » uses trees of 2048-4096 leaves,
- « Fast » uses trees of 256 leaves.

Tree Optimisation: One-tree technique

SDitH v2 - Design Rationale

- **MPC-in-the-Head paradigm**: possible existing frameworks
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VOLE-in-the-Head and **TC-in-the-Head**

(summer 2023) (fall 2023)

VOLE-in-the-Head (summer 2023)

- **Exponentially large fields:** e.g. $GF(2^{128})$ for L1
- **Only one protocol execution** (over the large field)
- **Based on 7-round protocol (or 5-round protocol)**
  - Rely on a **consistency check**
- Better signature sizes

TC-in-the-Head (fall 2023)

- **Small fields:** typically $GF(256), GF(2048), \dots$
- **Several parallel repetitions** (over the small field)
- **Based on 5-round protocol (or 3-round protocol)**

SDitH v2 - Design Rationale

- <u>**MPC-in-the-Head paradigm**</u>: possible existing frameworks
  - SDitH v1: rely on the framework using linear broadcast-based MPC
  - SDitH v2: ***two new MPCitH frameworks*** since the previous NIST deadline:

VOLE-in-the-Head (summer 2023) and **TC-in-the-Head** (fall 2023)

VOLE-in-the-Head (summer 2023)

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- **Based on 7-round protocol (or 5-round protocol)**
  - Rely on a **consistency check**
- ***Better signature sizes***

SDitH v2

TC-in-the-Head (fall 2023)

- **Small fields:** typically $GF(256), GF(256^2), \ldots$
- **Several parallel repetitions** (over the small field)
- **Based on 5-round protocol (or 3-round protocol)**

SDitH v2 - Design Rationale

- **Formalism**:
    - TCitH: sharing-based formalism
    - VOLEitH: VOLE-based formalism
    - SDitH v2: PIOP-based formalism (polynomial-based formalism)

🔎 *Motivation to choose the PIOP formalism:*

Simpler description of the scheme, that does not depend on MPC technology. Easier-to-understand scheme for those who do not already know those two frameworks.

[Fen24] Feneuil. *The Polynomial-IOP Vision of the Latest MPCitH Framework for Signature Schemes*. PQ Algebraic Cryptography Workshop, IHP 2024.

SDitH v2 - Underlying 3-Round Identification Scheme

Secret Key: $w \in \mathbb{F}_2^n$

Public Key: some degree-d multivariate polynomials $f_1, \ldots, f_m$ such that $f_1(w) = \ldots = f_m(w) = 0$.

① Sample n random degree-1 polynomials $P_1, \ldots, P_n$ such that $P_1(0) = x_1, \ldots, P_n(0) = x_n$.
Sample m random degree-$(d-1)$ polynomials $M_1, \ldots, M_m$.

② Commit to those polynomials.

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Sample m random degree-$(d-1)$ polynomials $M_1, \ldots, M_m$.

② Commit to those polynomials.

③ Send the polynomials $Q_1, \ldots, Q_m$ of degree at most $d-1$ defined such as

$$\begin{aligned} X \cdot Q_1(X) &:= X \cdot M_1(X) + f_1(P_1(X), \ldots, P_n(X)) \\ &\vdots \\ X \cdot Q_m(X) &:= X \cdot M_m(X) + f_m(P_1(X), \ldots, P_n(X)) \,. \end{aligned}$$

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① Sample n random degree-1 polynomials $P_1, \ldots, P_n$ such that $P_1(0) = x_1, \ldots, P_n(0) = x_n$.
Sample m random degree-$(d-1)$ polynomials $M_1, \ldots, M_m$.

② Commit to those polynomials.

③ Send the polynomials $Q_1, \ldots, Q_m$ of degree at most $d-1$ defined such as

$$\begin{aligned} X \cdot Q_1(X) &:= X \cdot M_1(X) + f_1(P_1(X), \ldots, P_n(X)) \\ &\vdots \\ X \cdot Q_m(X) &:= X \cdot M_m(X) + f_m(P_1(X), \ldots, P_n(X)) . \end{aligned}$$

④ Get a random evaluation point $r \in \mathscr{C}$ from the verifier.

⑤ Reveal the evaluations $P_1(r), \ldots, P_n(r)$ and $M_1(r), \ldots, M_m(r)$.

Soundness: $\dfrac{d}{|\mathscr{C}|}$ from the Schwartz-Zippel Lemma

SDitH v2 - Underlying 5-Round Identification Scheme

Secret Key: $w \in \mathbb{F}_2^n$

Public Key: some degree-d multivariate polynomials $f_1, \ldots, f_m$ such that $f_1(w) = \ldots = f_m(w) = 0$.

① Sample n random degree-1 polynomials $P_1, \ldots, P_n$ such that $P_1(0) = x_1, \ldots, P_n(0) = x_n$.
Sample a random degree-$(d-1)$ polynomial M.

② Commit to those polynomials.

③ Get random coefficients $\gamma_1, \ldots, \gamma_n \in \mathbb{K}$ from the verifier.

④ Send the polynomial Q of degree at most $d-1$ defined such as

$$X \cdot Q(X) := X \cdot M(X) + \sum_{j=1}^{m} \gamma_j \cdot f_j(P_1(X), \ldots, P_n(X)).$$

⑤ Get a random evaluation point $r \in \mathscr{C}$ from the verifier.

⑥ Reveal the evaluations $P_1(r), \ldots, P_n(r)$ and $M(r)$.

Soundness: $\dfrac{d}{|\mathscr{C}|} + \dfrac{1}{|\mathbb{K}|}$

Round-2 Design Updates

Syndrome Decoding Arithmetization

SDitH v2 - Design Rationale

System of polynomial constraints:

given the constraints $f_1, \ldots, f_m$*, it is hard to find* w *such that* $f_1(w) = \ldots = f_m(w) = 0$.

$\Big\downarrow$ *VOLEitH (or TCitH)*

Signature scheme - SDitH

SDitH v2 - Design Rationale

Syndrome Decoding Problem:

given H and y, it is hard to find x such that
$y = Hx$ and x has at most w_H non-zero coordinates.

SD Arithmetization

System of polynomial constraints:

given the constraints $f_1, \ldots, f_m$, it is hard to find w such that $f_1(w) = \ldots = f_m(w) = 0$.

VOLEitH (or TCitH)

Signature scheme - SDitH

SDitH v2 - Design Rationale

Syndrome Decoding Problem:

given H and y, it is hard to find x such that
$y = Hx$ and x has at most w_H non-zero coordinates.

Using the SDitH v1 arithmetization:
- $|w| \approx 2100$
- $m \approx 14000$
- $d = 2$

👉 We would obtain sizes around 5.3 KB for short L1.

SD Arithmetization

System of polynomial constraints:

given the constraints $f_1, \ldots, f_m$, it is hard to find w such that $f_1(w) = \ldots = f_m(w) = 0$.

VOLEitH (or TCitH)

Signature scheme - *SDitH*

Parameters of the constraint system:

- *The witness size $|w|$*
- *The number m of constraints*
- *The degree d of the constraints*

SDitH v2 - Design Rationale

Syndrome Decoding Problem:

given H and y, it is hard to find x such that
$y = Hx$ and x has at most w_H non-zero coordinates.

SD Arithmetization

System of polynomial constraints:

given the constraints $f_1, \ldots, f_m$, it is hard to find w such that $f_1(w) = \ldots = f_m(w) = 0$.

VOLEitH (or TCitH)

Signature scheme - *SDitH*

Parameters of the constraint system:

- *The witness size $|w|$*
- *The number m of constraints*
- *The degree d of the constraints*

Using [BBGK24]:

- $|w| \approx 550$
- $m \approx 4100$
- $d = 12$

👉 We would obtain sizes around 4.9 KB for short L1.

[BBGK24] Bettaieb, Bidoux, Gaborit, Kulkarni. *Modelings for generic PoK and Applications: Shorter SD and PKP based Signatures*. ePrint 2024.

SDitH v2 - Design Rationale

Syndrome Decoding Problem:

given H and y, it is hard to find x such that
$y = Hx$ and x has at most w_H non-zero coordinates.

Provable reduction: SD → RSD
loss of $d \cdot \log_2 \frac{n}{w_H} - \log_2 \binom{n}{w_H}$ bits

Regular Syndrome Decoding Problem:

given H and y, it is hard to find $x := (v_1 \parallel v_2 \parallel \dots \parallel v_{w_H})$
such that $y = Hx$ and $\{v_i\}_i$ are elementary vectors
(have only one non-zero coordinate).

RSD Arithmetization

System of polynomial constraints:

given the constraints $f_1, \dots, f_m$, it is hard to
find w such that $f_1(w) = \dots = f_m(w) = 0$.

VOLEitH (or TCitH)

Signature scheme - SDitH

SDitH v2 - Design Rationale

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given H and y, it is hard to find x such that
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↓ *RSD Arithmetization*

System of polynomial constraints:

given the constraints $f_1, \dots, f_m$, it is hard to
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↓ *VOLEitH (or TCitH)*

Signature scheme - SDitH

SDitH v2 - Design Rationale

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RSD Arithmetization

System of polynomial constraints:

given the constraints $f_1, \ldots, f_m$, it is hard to
find w such that $f_1(w) = \ldots = f_m(w) = 0$.

VOLEitH (or TCitH)

Signature scheme - *SDitH*

⚠️ The security assumption in SDitH is still the **unstructured** SD problem! The regular SD problem is *just* a proof artifact.

SDitH v2 - Design Rationale

Syndrome Decoding Problem:

given H and y, it is hard to find x such that
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VOLEitH (or TCitH)

Signature scheme - SDitH

⚠️ The security assumption in SDitH is still the **unstructured** SD problem! The regular SD problem is *just* a proof artifact.

(R)SD Arithmetization

[OTX24] Ouyang, Tang, Xu. *Code-Based Zero-Knowledge from VOLE-in-the-Head and Their Applications: Simpler, Faster, and Smaller*. Asiacrypt 2024.

[BBGK24] Bettaieb, Bidoux, Gaborit, Kulkarni. *Modelings for generic PoK and Applications: Shorter SD and PKP based Signatures*. ePrint 2024.

We need to have a **compact representation for an elementary vector**

$$v = (0,0,\ldots,0,1,0,\ldots,0) \in \mathbb{F}_2^n$$

where the i^{th} coordinate is the non-zero one.

Let us denote $e_0 = (1,0)$ and $e_1 = (0,1)$. We have that

$$v = e_{b_0} \otimes e_{b_1} \otimes \ldots \otimes e_{b_{\ell-1}}$$

where $i := b_0 + 2 \cdot b_1 + \ldots + 2^{\ell-1} \cdot b_{\ell-1}$.

For example: when $n = 4$,

$$(1,0,0,0) = e_0 \otimes e_0 \qquad (0,0,1,0) = e_1 \otimes e_0$$

$$(0,1,0,0) = e_0 \otimes e_1 \qquad (0,0,0,1) = e_1 \otimes e_1$$

(R)SD Arithmetization

[OTX24] Ouyang, Tang, Xu. *Code-Based Zero-Knowledge from VOLE-in-the-Head and Their Applications: Simpler, Faster, and Smaller*. Asiacrypt 2024.

[BBGK24] Bettaieb, Bidoux, Gaborit, Kulkarni. *Modelings for generic PoK and Applications: Shorter SD and PKP based Signatures*. ePrint 2024.

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where $i := b_0 + 2 \cdot b_1 + \ldots + 2^{\ell-1} \cdot b_{\ell-1}$.

RSD Arithmetization: Constraints in $y = Hx$ for $x := (v_1 \parallel \ldots \parallel v_{w_H})$, when writing all v_i as a tensorial product of elementary vectors.

Constraint Degree = 8

(R)SD Arithmetization

Relaxed arithmetization used in SDitH v2:
Better signature sizes, better computation performance

We need to have a **compact representation for an elementary vector**

$$v = (0,0,\ldots,0,1,0,\ldots,0) \in \mathbb{F}_2^n$$

where the i^{th} coordinate is the non-zero one.

Let us denote $e_j^{(\mu)} \in \mathbb{F}_2^{\mu}$ the j^{th} elementary vector. We have that

$$v = e_{c_0}^{(\mu_0)} \otimes e_{c_1}^{(\mu_1)} \otimes e_{c_2}^{(\mu_2)} \otimes e_{c_3}^{(\mu_3)}$$

where $i := c_0 + \mu_0 \cdot c_1 + (\mu_1\mu_0) \cdot c_2 + (\mu_3\mu_1\mu_0) \cdot c_2$, with $c_j \in \{0,\ldots,\mu_j - 1\}$.

RSD Arithmetization: Constraints in $y = Hx$ for $x := (v_1 \parallel \ldots \parallel v_{w_H})$,
when writing all v_i as a tensorial product of elementary vectors.

Constraint Degree = 4

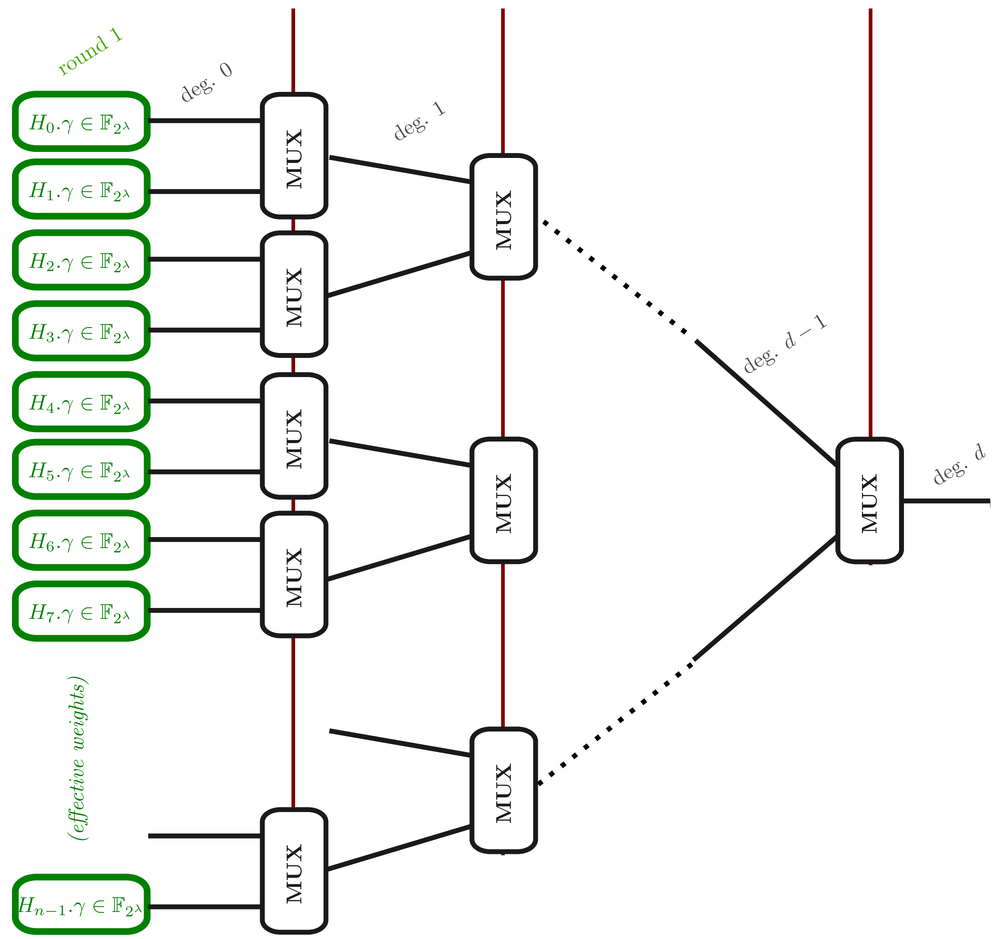
Round-2 Design Updates

Optimizations

SDitH v2 - Design Rationale

Used Optimizations (Arithmetic):

- Computation Speed-Up using Selection/Mux Tree



- Folding: using Gray code

👉 *See SDitH's specifications for details.*

SDitH v2 - Design Rationale

Used Optimizations (Symmetric primitives):

- Tree optimisation: one-tree technique [BBM$^+$24]

  [BBM$^+$24] Baum, Beullens, Mukherjee, Orsini, Ramacher, Rechberger, Roy, Scholl. *One tree to rule them all: optimizing GGM trees and OWFs for post-quantum signatures.* Asiacrypt 2024.

- PRG: based on AES-128 and Rijndael-256, instead of SHAKE
- Seed Commitment: based on AES-128 and Rijndael-256, instead of SHAKE

👉 *See SDitH's specifications for details.*

Round-2 Performance Updates

Performance on AVX-based CPU

We propose an optimized implementations for AVX2-based CPU.

SDitHv2 Instance		***PK Size***	***Sig. Size***	***Sig. Running time***	***Verif. Running time***
NIST I	Short	70 B	3 705 B	≈ 31.7 Mcycles	≈ 27.2 Mcycles
	Fast		4 484 B	≈ 9.3 Mcycles	≈ 8.2 Mcycles
NIST III	Short	98 B	7 964 B	≈ 189.7 Mcycles	≈ 176.6 Mcycles
	Fast		9 916 B	≈ 28.9 Mcycles	≈ 25.9 Mcycles
NIST V	Short	132 B	14 121 B	≈ 271.6 Mcycles	≈ 254.4 Mcycles
	Fast		17 540 B	≈ 43.4 Mcycles	≈ 39.5 Mcycles

Benchmark from PQ-Sort (https://pqsort.tii.ae)

Signature size: saving between 56% and 61%

Advantages & Limitations

Advantages & Limitations

- **Advantages**:
  - Conservative assumption: based on the oldest code-based hard problem

    The ***unstructured binary*** syndrome decoding problem
  - Adaptive and tunable parameters
  - Competitive code-based signatures: 3.7 KB for L1
  - Very small public keys: around 120~240 bytes
  - Competitive signature + public key size: for L1 short, 3.8 KB
    - 3.7 KB for ML-DSA, and 7.8 for SLH-DSA

- **Limitations**:
  - Quadratic growth w.r.t. to the security level
  - Limited performance: slow compared to lattice-based schemes, but competitive with many other post-quantum signature schemes.